AV-8213

M.A./M.Sc (IIIrd Semester) Examination

2015-16

Mathematics - Integration Theory

Model Answers

1 (i) Signed measure: A set function ν defined on a measurable space $(X, \mathcal{A})$ is said to be a signed measure if

(i) ν takes atmost one of the values $+\infty$ and $-\infty$

(ii) $\nu(\emptyset)=0$ and

(iii) $\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$ if $E_n \cap E_m = \emptyset$ for $n \neq m$

(ii) Suppose $F \in \mathcal{A}$ and $F \subset E$, then for any $A \subset F$ with $A \in \mathcal{A}$ we have $A \subset E$ so that $\nu(A) \leq 0$. That is, every measurable subset F of a negative set E is also a negative set.

(iii) Since $B = B - A + A$, $(B-A) \cap A = \emptyset$.

Hence $\nu(B) = \nu(B-A) + \nu(A)$

Hence if $\nu(B-A)$ and $\nu(A)$ both are ∞, then $\nu(B)$ is ∞.

If $\nu(B-A)$ or $\nu(B)$ is ∞, then again $\nu(B)$ is ∞. Given $|\nu(B)| < \infty$, Hence $\nu(B)$ will be finite if both $\nu(B-A)$ or $\nu(A)$ both are finite. Hence $|\nu(A)| < \infty$.

(iv) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let ν be a measure defined on $\mathcal{A}$ s.t ν is absolutely continuous w.r.t μ. Then there exists a non-negative measurable function f s.t

$$\nu(E) = \int_E f \, d\mu \quad \forall \; E \in \mathcal{A}.$$

The function f is unique in the sense that if g is any measurable function with this property, then $g(x) = f(x)$ almost everywhere in X w.r.t μ

(v) Suppose $E = [a,b] \times [c,d]$. Also suppose that f is Lebesgue integrable over E. Then

$$\iint_E f(x,y)\, dx\, dy = \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx \quad \text{and}$$

$$\iint_E f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy.$$

(vi) $y \in (\tilde{E})_x \iff (x,y) \in \tilde{E}$

$\iff (x,y) \in X \times Y$ and $(x,y) \notin E$

$\iff y \in Y$ and $y \notin E_x$, since $X_x = x$

$\iff y \in (\tilde{E_x})$

(vii) Since $(\mu \times \nu)(E) = 0$, there exists a set $F \in \mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and $(\mu \times \nu)(F) = 0$, it follows that $\nu(F_x) = 0$ for almost all x. Now since $E_x \subset F_x$ and ν is complete we get that $\nu(E_x) = 0$ for almost all x.

(viii) It is sufficient to prove that $\int_0^\infty \left|\frac{\sin x}{x}\right| = \infty$

For this, consider the integral

$$\int_0^{n\pi} \frac{|\sin x|}{x}\,dx = \sum_{k=1}^{\infty} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx$$

$$= \sum_{k=1}^{n} \int_0^{\pi} \frac{|\sin\{z+(k-1)\pi\}|}{z+(k-1)\pi}\,dz$$

$$\geq \sum_{k=1}^{n} \int_0^{\pi} \frac{|\sin(z+(k-1)\pi)|}{k\pi}\,dz$$

$$= \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{k} \int_0^{\pi} |\sin z|\,dz$$

$$= \sum_{k=1}^{n} \frac{2}{k\pi}$$

Thus, $\lim_{n\to\infty} \int_0^{n\pi} \frac{|\sin x|}{x}\,dx \geq \sum_{k=1}^{\infty} \frac{2}{k\pi} = \infty \Rightarrow \int_0^{\infty} \frac{|\sin x|}{x}\,dx = \infty$.

(ix) Consider the ~~fn~~ $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The function f is clearly bounded. To show that f is not of bounded variation, consider a partition $\left\{1, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots, \frac{2}{2n+1}, 0\right\}$ of $[0,1]$, n being a positive integer. For this partition, we have

$$\left|f(1) - f\left(\tfrac{2}{3}\right)\right| + \cdots + \left|f\left(\tfrac{2}{2n+1}\right) - f(0)\right| = \frac{2}{3} + \left(\frac{2}{3} + \frac{2}{5}\right) + \cdots + \left(\frac{2}{2n+1}\right)$$

$$= 4\left(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}\right)$$

Since $\Sigma \frac{1}{2n+1}$ is dgt, its partial sum $\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n+1}\right)$ is not bounded above, Hence $T_0^1(f) = \infty$. (or)

Any suitable example is also consider

(x) A measure μ defined on the σ-algebra of Baire sets is called a Baire measure if it is finite for each compact Baire set.

A measure is called regular if it is outer and inner regular (outer & inner regular definitions are also needed to write).

Q.2 If E itself is a negative set then we may take $A = E$ and the theorem is proved.

Now consider the case in which E itself is not a negative set. Then E must contain a subset of positive measure.

This $\Rightarrow \exists\, E_1 \subset E$ and a smallest positive integer n_1 s.t.

$$\nu(E_1) > \frac{1}{n_1}$$

Evidently $E = (E-E_1) \cup E_1$ and $(E-E_1) \cap E_1 = \phi$

Hence $\nu(E) = \nu(E-E_1) + \nu(E_1)$ — ①

(or) $\nu(E-E_1) = \nu(E) - \nu(E_1)$ — ②

Since $\nu(E) < \infty$ is finite and hence ① declares that $\nu(E-E_1)$ and $\nu(E_1)$ both are finite.

Since $\nu(E) < 0$. Hence $\nu(E)$ is a negative finite number. Then ② shows that $\nu(E-E_1) < 0$.

If $E-E_1$ is a negative set then we may take $A = E-E_1$. Otherwise $E-E_1$ contains subsets of positive measure. Let n_2 be a smallest positive number s.t there is a measurable set $E_2 \subset E-E_1$ with the property that $\nu(E_2) > \frac{1}{n_2}$

$\because E = [E-E_1 \cup E_2] \cup (E_1 \cup E_2)$, $(E-E_1 \cup E_2) \cap (E_1 \cap E_2) = \phi$

$$\nu(E) = \nu(E-E_1 \cup E_2) + \nu(E_1) + \nu(E_2)$$
$$= \nu(E-E_1 \cup E_2) + \nu(E_1 \cup E_2)$$

or $\quad \nu(E-E_1 \cup E_2) = \nu(E) - \nu(E_1) - \nu(E_2) < 0$

For $\nu(E) < 0$, $\nu(E_n) > 0$ for $n = 1, 2, \ldots$

Thus $\quad \nu(E-E_1 \cup E_2) < 0$

This shows that $E-E_1 \cup E_2$ is a set of negative measure. If $E-E_1 \cup E_2$ is a negative set, then we may takes $A = E - E_1 \cup E_2$. Otherwise we repeat the above process.

Continuing this process, we shal set either a negative subset A of E s.t. $\nu(A) < 0$ or a seq. $\langle n_k : k \in N \rangle$ of positive integers and a seq. $\langle E_k \rangle$ of distinct measurable sets s.t.

$$\frac{1}{n_k} < \nu(E_k) < \infty$$

In this latter case, suppose that

$$A = (E - \cup E_i) \quad -③$$

As usual, from this equation we can deduce that

$$\nu(E) = \nu(A) + \nu(\cup E_i) = \nu(A) + \sum \nu(E_k)$$
$$> \nu(A) + \sum \frac{1}{n_k} \quad -④$$

since ν assumes atmost one of values $-\infty$ and ∞ and $\nu(E)$ is finite, therefore ④ says that $\nu(A)$ is finite and the series

$$\sum \frac{1}{n_k} \text{ is cgt}$$

From ④, $\nu(A) < \nu(E) - \sum \frac{1}{n_k}$

$= \text{a negative} ~~no~~ \Rightarrow \nu(A) < 0.$

③ says that A is a measurable set.

For an enumerable union of measurable sets is measurable, and difference of two measurable sets is measurable.

It remains to show that A is a negative set.

For this let $B \subset A$ be any arbitrary measurable set.

$$B \subset A = E - \bigcup_{k=1}^{\infty} E_k \subset E - \bigcup_{k=1}^{i-1} E_k \Rightarrow B \subset E - \bigcup_{k=1}^{i-1} E_k$$

This choice of +ve integer shows that $\nu(B) \le \frac{1}{n_i - 1} \quad -⑤$

making $n_i \to \infty$ in ⑤, we get $\nu(B) \le 0$.

Thus we have shown that A is a measurable set s.t $\nu(A) < 0$ and ~~$\nu(B)$~~ $\forall B \subset A$ s.t B is a measurable set $\Rightarrow \nu(B) \le 0$.

This proves that A is a negative set with $\nu(A) < 0$.

(b) Suppose ν is a signed measure on a measurable space $(X, \mathcal{A})$. Then $\exists$ a positive set P and a negative set Q s.t. $P \cap Q = \emptyset$. $X = P \cup Q$.

<u>Proof</u>: Let $\mathcal{A}$ be a σ-algebra of subsets of X. Let ν be a signed measure on a measurable space $(X, \mathcal{A})$. Since ν assumes atmost one of the values $-\infty$ and ∞. without loss of generality we can suppose that ν doesnot take $-\infty$. Consider the family $\mathcal{B}$ of all negative subsets of X and let $\lambda = \inf\{\nu(E) : E \in \mathcal{B}\} \quad -①$

Then $\exists$ a sequence $\langle E_n \rangle$ in $\mathcal{B}$ s.t $\lim_{n\to\infty} \nu(E_n) = \lambda$.

$\mathcal{B}$ is a family of negative sets

$\Rightarrow$ $\langle E_n \rangle$ is a ~~seq.~~ of negative sets

$\Rightarrow$ $\cup E_i$ is a negative set,

$\Rightarrow$ Q is a negative set on taking $Q = \cup E_i$

Thus Q is a negative subset X. Then according to ①, $\nu(Q) \geq \lambda$.

Next we consider $Q - E_n$ of Q.

$\because$ $Q = (Q - E_n) \cup E_n$

$\therefore$ $\nu(Q) = \nu(Q - E_n) + \nu(E_n) \leq \nu(E_n) \quad \forall n \in \mathbb{N}$

From ①, these two facts prove that $\nu(Q) \leq \lambda$.

Thus we have shown that $\nu(Q) \leq \lambda$ and $\nu(Q) \geq \lambda$

This $\nu(Q) = \lambda \Rightarrow \lambda > -\infty$.

Next our aim is to show that $P = X - Q$ is a positive subset of X. Suppose the contrary. Then P is not positive and so P is negative. Hence by definition, for every measurable set $E \subset P$, $\nu(E) < 0$. Now E is a measurable subset of X with negative measure. making use of the result: $-\infty < \nu(E) < 0$, then E contains a set A with $\nu(A) < 0$, we obtain a negative set $A \subset E$ s.t $\nu(A) < 0$.

Since A and Q both are disjoint negative subsets of X and their union $A \cup Q$ is also negative. Consequently

$\nu(A \cup Q) \geq \lambda$, by ①

But $\lambda \leq \nu(A \cup Q) = \nu(A) + \nu(Q) = \nu(A) + \lambda$

$\lambda \leq \nu(A) + \lambda \Rightarrow \nu(A) \geq 0$

Contrary to the fact that $\nu(A) < 0$.

Hence our assumption is wrong. Therefore P is +ve

Thus $P = X - Q$ is +ve and Q is −ve $\Rightarrow X = P \cup Q$, $P \cap Q = \emptyset$.

Q.3 (a) Since f is integrable w.r.t μ on X, we get $\int_E f^+ d\mu$ or $\int_E f^- d\mu$ are both finite so that $\phi(E) = \int_E f^+ d\mu - \int_E f^- d\mu$ is ∞ or $-\infty$ only but not both for $E \in A$. That is (i) of signed measure definition holds for ϕ.

clearly $\varphi(\emptyset) = \int_\emptyset f d\mu = 0$ giving the condition ii of signed measure for the function ϕ.

Suppose $\{E_n\}$ is a pairwise disjoint set in A and $E = \cup E_n$

Then
$$\varphi(\cup E_n) = \int_E f^+ d\mu - \int_E f^- d\mu$$
$$= \sum_{n=1}^{\infty} \int_{E_n} f^+ d\mu - \sum_{n=1}^{\infty} \int_E f^- d\mu$$

implies that
$$\varphi(\cup E_n) = \sum \left(\int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right)$$
$$= \sum \int_{E_n} f \, d\mu = \sum \varphi(E_n)$$

Then φ is a signed measure on (X, A)

(b) Let (X, A, μ) be a σ-finite measure space and ν be a σ-finite measure defined on A. Then $\exists$ two uniquely determined measures ν_0 and ν_1 s.t $\nu = \nu_0 + \nu_1$, $\nu_0 \perp \mu$, $\nu_1 \ll \mu$.

Proof: $\lambda = \mu + \nu$. μ and ν are σ-finite implies that λ is σ-finite.

Evidently $\mu \ll \lambda$ and $\nu \ll \lambda$.

$\mu \ll \lambda$ stands for 'μ is absolutely continuous w.r.t λ"

By Radon-Nikodyn Theorem, we are able to find non-negative function $f, g : X \to [-\infty, \infty]$ s.t
$$\mu(E) = \int_E f d\lambda, \quad \nu(E) = \int_E g \, d\lambda, \ \forall E \in A.$$

Let $A = \{x \in X : f(x) > 0\}$, $B = \{x \in X : f(x) = 0\}$

Then $X = A \cup B$, $A \cap B = \emptyset$, Furthermore $\mu(B) = \int_B f d\lambda = 0$.

Define two functions $\nu_0, \nu_1 : A \to [0, \infty)$ by requiring that
$$\nu_0(E) = \nu(E \cap B), \quad \nu_1(E) = \nu(E \cap A) \ \forall E \in A.$$

Then ν_0 and ν_1 are measures on $\mathcal{A}$ and satisfy the condition

$$\nu = \nu_0 + \nu_1$$

$$\nu_0(A) = \nu(A \cap B) = \nu(\emptyset) = 0, \text{ or } \nu_0(A) = 0$$

Thus $\mu(B) = 0 = \nu_0(A) = \nu_0(X - B)$

$\mu(B) = 0 = \nu_0(X - B)$

This $\Rightarrow$ ν_0 is mutually singular to μ

$\Rightarrow$ $\nu_0 \perp \mu$.

To show that $\nu_1 \ll \mu$.

For this let $E \in \mathcal{A}$ be arbitrary s.t $\mu(E) = 0$

Then $\int_E f \, d\lambda = \mu(E) = 0$ or $\int_E f \, d\lambda = 0$.

Also $f(x) \geq 0 \ \forall x \in E$.

This $f = 0$ a.e on E relative to λ

since $f \geq 0$ on $A \cap E$ and hence $\nu_1(E) = \nu(E \cap A)$ by definition of ν_1

$\leq \theta \lambda(E \cap A) = 0$

$\therefore$ $\nu_1(E) \leq 0$. But $\nu_1(E) \geq 0$

Combining these two results $\nu_1(E) = 0$.

Thus we have shown that $\mu(E) = 0 \Rightarrow \nu_1(E) = 0$

$\Rightarrow$ $\nu_1 \ll \mu$.

Uniqueness: To show that ν_0 and ν_1 are unique, let ν_0' and ν_1' be measures s.t $\nu = \nu_0' + \nu_1'$ and has the same property as that of the measures ν_0 and ν_1 respectively. Then $\nu = \nu_0 + \nu_1$ and $\nu = \nu_0' + \nu_1'$ are the two Lebesgue decomposition of ν. Then $\nu_0 - \nu_0' = \nu_1' - \nu_1$. Again $\nu_1' - \nu_1$ is absolutely continuous and $\nu_0 - \nu_0'$ is singular relative to ν. We have $\nu_0 = \nu_0'$, $\nu_1 = \nu_1'$.

Q.4. A bounded fn. f is Lebesgue integrable over E. Then

$$\inf_{\psi \geq f} \int_E \psi(x)\,dx = \sup_{\varphi \leq f} \int_E \varphi(x)\,dx = I \text{ (say)}$$

for all simple functions φ and ψ

Given an integer n, $\exists$ a simple fn. φ_n and ψ_n such that $\varphi_n(x) \leq f(x) \leq \psi_n(x)$ satisfying

$$\int_E \psi_n(x)\,dx < I + \frac{1}{2n} \text{ and } \int_E \varphi_n(x)\,dx > I - \frac{1}{2n}$$

This gives

$$\int_E \psi_n(x)\,dx - \int_E \varphi_n(x)\,dx < \frac{1}{n} \quad \text{—} \; (1)$$

Define the fn's $\psi^* = \inf \psi_n$ and $\varphi^* = \sup \varphi_n$.

Since for each n, φ_n and ψ_n are measurable fn's, the fn φ^* and ψ^* are measurable and $\varphi^*(x) \leq f(x) \leq \psi^*(x)$. Now consider the sets

$$\Delta = \{x : \varphi^*(x) < \psi^*(x)\}$$

$$\Delta_\nu = \{x : \varphi^*(x) < \psi^*(x) - \tfrac{1}{\nu}\}$$

$$\Delta_{\nu,n} = \{x : \varphi_n(x) < \psi_n(x) - \tfrac{1}{\nu}\}$$

we note that

(a) $\Delta = \bigcup_{\nu=1}^{\infty} \Delta_\nu$

(b) $\Delta_\nu \subset \Delta_{\nu,n}, \ \forall n$

(c) $m(\Delta_{\nu,n}) < \nu/n$; for if $m(\Delta_{\nu,n}) \geq \nu/n$, then

$$\int_{\Delta_{\nu,n}} \psi_n(x)\,dx - \int_{\Delta_{\nu,n}} \varphi_n(x)\,dx = \int_{\Delta_{\nu,n}} \{\psi_n(x) - \varphi_n(x)\}\,dx$$

$$\geq \frac{1}{\nu} m(\Delta_{\nu,n}) \geq \frac{1}{n},$$

which contradicts (1).

Since n is arbitrary, we, in view of (b) and (c) above, have $m(\Delta_\nu) = 0$ and hence $m(\Delta) = 0$. This proves that $\varphi^* \geq \psi^*$ a.e. But $\varphi^* \leq \psi^*$. Hence $\varphi^* = \psi^* = f$ a.e., and since each of the fn φ^* and ψ^* is measurable, the fn f is measurable.

On the other hand, assume that f is a measurable fn on E. Suppose f is bounded by M. Then

$$-M \leq f(x) \leq M, \quad \forall x \in E$$

Divide the interval $[-M, M]$ into $2n$ equal parts and consider the sets $E_k = \{x \in E : (k-1)\frac{M}{n} < f(x) \leq k\frac{M}{n}\}, \quad -n \leq k \leq n.$

Clearly, $\{E_k : -n \leq k \leq n\}$ is a countable collection of pairwise disjoint measurable sets such that $E = \bigcup E_k$. Therefore

$$m(E) = \sum_{k=-n}^{n} m(E_k)$$

For each n, if we define simple fn ψ_n and φ_n as

$$\psi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} k \chi_{E_k}(x)$$

and
$$\varphi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} (k-1) \chi_{E_k}(x)$$

then they satisfy $\varphi_n(x) \le f(x) \le \psi_n(x)$. Thus

$$\begin{cases} \inf_{\psi \ge f} \int_E \psi(x)\,dx \le \int_E \psi_n(x)\,dx = \frac{M}{n} \sum_{k=-n}^{n} k\, m(E_k) \\ \sup_{\varphi \le f} \int_E \varphi(x)\,dx \ge \int_E \varphi_n(x)\,dx = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\, m(E_k) \end{cases}$$

$$\Rightarrow \inf_{\psi \ge f} \int_E \psi(x)\,dx - \sup_{\varphi \le f} \int_E \varphi(x)\,dx \le \frac{M}{n} \sum_{k=-n}^{n} m(E_k) = \frac{M}{n} m(E)$$

Since n is arbitrary, we have

$$0 \le \inf_{\psi \ge f} \int_E \psi(x)\,dx - \sup_{\varphi \le f} \int_E \varphi(x)\,dx \le 0.$$

Hence f is Lebesgue integrable on E.

Q. 5 (a) Let $\{f_n\}$ be a sequence of nonnegative measurable fns and $f_n \to f$ a.e on E. Then

$$\int_E f \le \lim_{n\to\infty} \int_E f_n.$$

Proof: we may assume, without any loss of generality, that the seq $\{f_n\}$ converges to f everywhere on E since the integrals over sets of measure zero are zero.

Let h be bounded measurable function such that $h \le f$ vanishes outside a set of finite measure.

$$m(\{x \in E : h(x) \ne 0\}) < \infty.$$

Let us denote this set by E'. Define a seq $\{h_n\}$ of function by setting

$$h_n(x) = \min\{h(x), f_n(x)\}.$$

Then each $|h_n|$ is clearly bounded by the bound of $|h|$ and vanishes outside E'. Moreover, we denote that

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \min\{h(x), f_n(x)\}$$
$$= \min\{h(x), f(x)\}$$
$$= h(x) \quad x \in E'$$

Thus $\{h_n\}$ is a uniformly bounded sequence of measurable functions such that $h_n \to h$ on E'. Therefore, by the Bounded Convergence Theorem we have

$$\lim_{n\to\infty} \int_{E'} h_n = \int_{E'} h.$$

$$\Rightarrow \int_E h = \int_{E'} h = \lim_{n\to\infty} \int_{E'} h_n \le \lim_{n\to\infty} \int_{E} f_n.$$

Hence, taking the supremum over all $h \le f$, we get

$$\int_E f \le \lim_{n\to\infty} \int_E f_n.$$

(b) Corresponding to each positive integer n, define

$h_n = \inf\{f_\nu : \nu \ge n\}$. Each h_n is measurable.

Since $h_n \le f_n$, for $\nu \ge n$ we have

$$\int_E h_n \le \int_E f_\nu, \quad \text{for } \nu \ge n$$

$$\Rightarrow \int_E h_n \le \lim_{\nu\to\infty} \int_E f_\nu, \quad \forall m \in N$$

One may verify that $\{h_n\}$ is an increasing ~~sq~~ which converges to the limit ~~of~~ h. Therefore, we get

$$\int_E \lim_{n\to\infty} f_n = \int_E h = \lim_{n\to\infty} \int_E h_n \le \lim_{n\to\infty} \int_E f_n$$

by using the Theorem monotone convergence Theorem.

6(a) Let f be a bounded variation on $[a,b]$. Define g and h by $g = \frac{1}{2}(V_f + f)$ and $h = \frac{1}{2}(V_f - f)$, so that $f = g - h$.

Now, if x_1, x_2 is any pair of points in $[a,b]$ with $x_2 > x_1$, then

$$g(x_2) - g(x_1) = \frac{1}{2}\left[\{V_f(x_2) - V_f(x_1)\} + \{f x_2 - f x_1\}\right]$$
$$h(x_2) - h(x_1) = \frac{1}{2}\left[\{V_f x_2 - V_f x_1\} - \{f x_2 - f x_1\}\right]$$

But, f being of bounded variation on $[a,b]$, in particular on $[x_1,x_2]$ we have

$$|fx_2 - fx_1| \le T_{x_1}^{x_2}(f) = \nu_f(x_2) - \nu_f(x_1).$$

Hence $gx_2 \ge gx_1$ and $hx_2 \ge hx_1$ verifying that g and h both are monotone increasing functions.

Conversely, if $f = g - h$ where g and h are monotone increasing fns, then for any partition P of $[a,b]$, we have

$$\sum_{i=1}^{n} |fx_i - fx_{i-1}| \le \sum_{i=1}^{n} [gx_i - gx_{i-1}] + \sum_{i=1}^{n} [hx_i - hx_{i-1}]$$
$$= g(b) - g(a) + h(b) - h(a)$$
$$= T_a^b(f) < \infty$$

Hence f is a fn. of bounded variation.

(6) Let if possible, $f \ne 0$ a.e. in $[a,b]$. Suppose $f(t) > 0$ on a set E of positive measure. Then there exists a closed set $F \subset E$ with $m(F) > 0$. Put $A = (a,b) - F$. Then A is an open set and we have

$$0 = \int_a^b f(t)\,dt = \int_{A \cup F} f(t)\,dt = \int_A f(t)\,dt + \int_F f(t)\,dt$$

$$\Rightarrow \quad \int_A f(t)\,dt = -\int_F f(t)\,dt.$$

But $f(t) > 0$ on F with $m(F) > 0$ implies that $\int_F f(t)\,dt \ne 0$

Therefore $\int_A f(t)\,dt \ne 0$.

Now, A being an open set, it can be expressed as a union of countable collection $\{(a_n, b_n)\}$ of disjoint open intervals. Then

$$0 \ne \int_A f(t)\,dt = \sum_n \int_{a_n}^{b_n} f(t)\,dt$$

$$\Rightarrow \quad \int_{a_n}^{b_n} f(t)\,dt \ne 0 \text{ for some } n$$

$$\Rightarrow \quad \text{either } \int_a^{a_n} f(t)\,dt \ne 0 \text{ or } \int_a^{b_n} f(t)\,dt \ne 0.$$

In either case, we see that if f is positive on a set of positive measure, then for some $x \in [a,b]$ we have

$$\int_a^x f(t)\,dt \neq 0.$$

Similarly assertion ~~d~~ is obtained if f is negative on a set of positive ~~&~~ measure. Hence the result follows by contradiction.

7(a) We may assume, without any loss of generality, that $f \geq 0$. Define a ~~seq.~~ $\{f_n\}$ of functions $f_n : [a,b] \to \mathbb{R}$, where

$$f_n(x) = \begin{cases} fx & \text{if } fx \leq n \\ n & \text{if } fx > n. \end{cases}$$

Clearly, each f_n is a bounded and measurable ~~fn.~~ and so, ~~by~~ we get.

$$\frac{d}{dx}\int_a^x f_n = f_n(x) \text{ a.e.}$$

Also, $f - f_n \geq 0$ for all n, and therefore, the ~~fn.~~ Gn defined by

$$G_n(x) = \int_a^x f - f_n$$

is an increasing ~~fn.~~ of x, which must have a derivative almost everywhere. and this derivative would, clearly, be non-negative.

Thus, the relation

$$F(x) = \int_a^x f(t)\,dt + F(a) = G_n(x) + \int_a^x f_n(t)\,dt + F(a)$$

it follows that

$$F'(x) = G_n'(x) + f_n(x) \text{ a.e.}$$
$$\geq f_n(x) \text{ a.e } \quad \forall n$$

Since n is arbitrary, we have

$$F'(x) \geq f(x) \text{ a.e}$$

$$\Rightarrow \int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx = F(b) - F(a)$$

Therefore,

$$\int_a^b F'(x)\,dx = F(b) - F(a) = \int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b [F'(x) - f(x)]\,dx = 0 \text{. Since } F'(x) - f(x) \geq 0 \text{ a.e.,}$$

this ~~gives~~ gives that $F'(x) - f(x) = 0$ a.e., and so $F'(x) = f(x)$ a.e.

(b) For $\varepsilon = 1$, there is a $\delta > 0$ such that for every finite collection $\{(x_i, x_i')\}_{i=1,2,\dots}$ of pairwise disjoint intervals in $[a,b]$.
with $\sum |x_i' - x_i| < \delta$, we have

$$\sum |f x_i' - f x_i| < 1$$

select a natural no. $N > \frac{b-a}{\delta}$.

Divide $[a,b]$ by means of points

$$a = c_0 < c_1 < c_2 < \dots < c_N = b$$

such that $c_j - c_{j-1}$, for $j = 1, 2, \dots, N$. Therefore, for every finite collection $\{(x_i, x_i')\}$ of pairwise disjoint intervals in $[c_{j-1}, c_j]$, we have

$$\sum |f x_i' - f x_i| < 1$$

$$\Rightarrow \quad T_{c_{j-1}}^{c_j}(f) \le 1, \quad j = 1, 2, \dots, N.$$

Hence $\quad T_a^b(f) = \sum_{j=1}^{N} T_{c_{j-1}}^{c_j}(f) \le N < \infty.$

and this proves that f is a fn. of bounded variation.

8 The space $C_c(X)$ is a vector lattice, and I will be Daniell integral if it satisfies the Daniell Condition D. let $\langle \varphi_n \rangle$ be a ~~seq.~~ of fn.s in $C_c(X)$ which decrease to zero and let K be the support of φ_1. Let φ be a non-negative fn. in $C_c(X)$ which is positive on K. Then for a given $\varepsilon > 0$ the sets $F_n = \{x : \varphi_n(x) \ge \varepsilon \varphi(x)\}$ are a decreasing family of closed subsets of K whose intersection is empty. Thus for some N we have $F_N = 0$, and $\varphi_n < \varphi \varepsilon$ for $n \ge N$. Thus $I(\varphi_n) \le \varepsilon I(\varphi)$ for $n \ge N$, and $I(\varphi_n) \to 0$, since ε was arbitrary. The existence of a Baire measure μ so that $I(f) = \int f d\mu$ for each $f \in C_c(X)$ now follows from the Stone th. To ~~see~~ the uniqueness of μ, we note that, if K is a compact G_δ, then χ_K is the limit of a decreasing seq. $\langle \varphi_n \rangle$ of fn. in $C_c(X)$. Since φ_1 is integrable, the Lebesgue Convergence

theorem asserts that

$$\mu K = \lim \int \varphi_n \, d\mu = \lim I(\varphi_n).$$

Thus μK is uniquely determined by I. By the measure of μ is uniquely determined on the σ-bounded Baire sets and hence on all Baire sets if X is compact

——— X ———